A TOPOLOGICAL AND GEOMETRIC APPROACH TO FIXED POINTS RESULTS FOR SUM OF OPERATORS AND APPLICATIONS

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Abstract. In this paper we establish a fixed point result of Krasnoselskii type for the sum $A+B$, where A and B are continuous maps acting on locally convex spaces. Our results extend previous ones. We apply such results to obtain strong solutions for some quasi-linear elliptic equations with lack of compactness. We also provide an application to the existence and regularity theory of solutions to a nonlinear integral equation modeled in a Banach space. In the last section we develop a sequentially weak continuity result for a class of operators acting on vector-valued Lebesgue spaces. Such a result is used together with a geometric condition as the main tool to provide an existence theory for nonlinear integral equations in $L^p(E)$.

1. Introduction

Many problems arising from the most diverse areas of natural science, when modeled under the mathematical point of view, involve the study of solutions of nonlinear equations of the form

$$Au + Bu = u, \quad u \in M, \tag{1.1}$$

where M is a closed and convex subset of a Banach space X, see for example [4, 6, 7, 8, 9, 10]. In particular, many problems in integral equations can be formulated in terms of (1.1). Krasnoselskii's fixed point Theorem appeared as a prototype for solving equations of the type (1.1), where A is a continuous and compact operator and B is, in some sense, a contraction mapping. Motivated by the observation that the inversion of a perturbed differential operator could yield a sum of a contraction and a compact operator, Krasnoselskii proved that the sum $A+B$ has a fixed point in M, if: **(i)** A is continuous and compact, **(ii)** B is a strict contraction and **(iii)** $Ax+By \in M$ for every $x, y \in M$. Since then a wide class of problems, for instance in integral equations and stability theory, have been contemplated by the Krasnoselskii fixed point approach. However, in several applications, the verification of **(iii)** is, in general, quite hard or even impossible to be done. As a tentative approach to grapple with such a difficulty, many interesting works have appeared in the direction of relaxing hypothesis **(iii)**.

In a recent paper [4], Burton proposes the following improvement for **(iii)**: (If $u = Bu + Av$ with $v \in M$, then $u \in M$). Subsequently, in [5], the following

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new asymptotic requirement was introduced:

$$\big(\text{If } \lambda \in (0,1) \text{ and } u = \lambda Bu + Av \text{ for some } v \in M, \text{ then } u \in M\big),$$

In this paper we explore another kind of generalization. Indeed we study suitable modifications on conditions **(i)** and **(ii)** as well. Notice that condition **(i)** involves continuity and compactness. Based on the well known fact that infinite dimensional Banach spaces are not locally compact, we suggest a locally convex topology approach to equation (1.1). The interpretation of some practical equation of the form (1.1) may face the problem that the operators involved are not even continuous. The freedom of choosing a more general notion of topology might remedy this difficulty. We should mention that others authors have already studied equation (1.1) in locally convex spaces [7, 23].

Condition **(ii)** is also, in same sense, quite restrictive. Indeed this condition implies norm-continuity. In this paper we shall suggest a much more general condition than strict contraction. All the generality of our results will be used in the applications. For instance, in section 6, when we shall be interested in proving optimal regularity of solutions to a nonlinear integral equation, we will apply our fixed point results on the space $W^{1,\infty}(I, E)$. The operators in question will be neither norm continuous nor weakly continuous. Thus the classical assumptions on fixed point results of Krasnoselskii type does not hold.

Our paper is organized as follows: In Section 2 we reformulate the Krasnoselskii fixed point Theorem for the locally convex setting. This new version of the Krasnoselskii fixed point Theorem we provide here generalizes, among others, the results in [5]. The main abstract results are introduced in this section: Theorem 2.2, Corollary 2.3 and Theorem 2.9.

In Section 3, we apply our fixed point theory to the solvability of one parameter operator equations of the form

$$Au + \lambda Bu = u, \;\; \lambda \geq 0.$$

In this section we shall restrict ourself to reflexive Banach spaces endowed with the weak topology. In the next section we exemplify the power of results established by studying an elliptic equation with lack of compactness. In particular we solve a quasi-linear elliptic equation with Sobolev critical exponent.

A nonlinear integral equation is studied in section 5. We provide an existence principle for the following nonlinear integral equation:

$$u(t) = f(u(t)) + \int_0^t g(s, u(s))ds, \quad u \in C(I, E), \tag{1.2}$$

where E is a reflexive Banach space and $I = [0, T]$. In section 6 we explore the optimal regularity of solutions of equation (1.2). The approach in this section explores all the generality of the results established in section 2. As part of our strategy to get Lipschitz regularity for solutions of equation (1.2), we suggest a new locally convex topology for the space $W^{1,\infty}(I, E)$. In such topology the ball of this space is compact. Furthermore, we can proof continuity of all the operators involved in equation (1.2). Such operators are not norm-continuous though.

Finally, in Section 7, we explore a new geometric idea of finding fixed point for sum of operators. Such an approach is motivated by a sort of strong triangular inequality for uniformly convex spaces. We apply this idea directly in the study of

the following challenging variant of (1.2):

$$u(t) = f(t, u(t)) + \Phi\left(t, \int_0^t k(t,s)u(s)ds\right), \quad u \in L^p(I, E), \tag{1.3}$$

where E is a uniformly convex space, $1 < p < \infty$ and $I = [0, T]$. As usual, the existence of solutions to (1.3) reduces to search fixed points for the operator $A + B$, where

$$\begin{aligned} Au(t) &= f(t, 0) + \Phi\left(t, \int_0^t k(t,s)u(s)ds\right), \\ Bu(t) &= f(t, u(t)) - f(t, 0). \end{aligned}$$

A geometric condition is used in order to assure that $(A + B)(B_{L^p(E)}(\overline{R})) \subseteq B_{L^p(E)}(\overline{R})$, for some $\overline{R} > 0$. Our final step toward the solution of equation (1.3) is a sequentially weak continuity result (Lemma 7.11) which guarantees the weak sequential continuity for the operator acting on vector-valued Lebesgue spaces.

2. Abstract Fixed Point Theorems for Sum of Operators

The notation and terminology used in this paper are standard. For convenience of the reader we recall some basic facts. Let $(X, \mathcal{T})$ be a Hausdorff locally convex topological vector space. The symbol $\mathcal{T}$ stands for the family of seminorms that generates the locally convex topology on X. In the sequel we will use the following well-known Schauder-Tychonoff fixed point Theorem for locally convex spaces.

Theorem 2.1. *Let M be a closed convex subset of a Hausdorff locally convex space X and let $T : M \to M$ be a continuous mapping such that $T(M)$ is relatively compact. Then T has a fixed point in M.*

Our first version of the Krasnoselskii fixed point Theorem is as follows.

Theorem 2.2. *Let M be a closed and convex subset of a Hausdorff locally convex space X and $A, B : M \to X$ be continuous operators such that*

(a) *$A(M)$ is relatively compact;*
(b) *$(I - B)$ is continuously invertible and $A(M) \subseteq (I - B)(M)$;*
(c) *If $u = B(u) + A(v)$ for some $v \in M$ then $u \in M$.*

Then $A + B$ has a fixed point in M.

Proof. Let us define $T\colon M \to M$ by

$$T(u) := (I - B)^{-1} \circ A(u).$$

T is well defined by item (a). The fact that T maps M into M follows by condition (c). Furthermore T is a continuous map and $T(M)$ is relatively compact. Applying Theorem 2.1 to the operator T we conclude there exists a $u \in M$ such that $T(u) = u$. Finally notice that a fixed point to T is actually a fixed point to $A + B$. □

Corollary 2.3. *Let M be a compact convex subset of a Hausdorff locally convex space X and $A, B : M \to X$ be continuous operators such that*

(a) *There exists a sequence $\lambda_n \to 1$, such that $(I - \lambda_n B)$ is injective and $A(M) \subseteq (I - \lambda_n B)(M)$, $\forall n$;*
(b) *For all $n \geq 1$, if $u = \lambda_n B(u) + A(v)$ for some $v \in M$ then $u \in M$.*

Then $A + B$ has a fixed point in M.

Proof. We first notice that once M is compact and B is continuous, we automatically have the continuity of $(I-\lambda_n B)^{-1}$ and hence of the operators $T_n := (I-\lambda_n B)^{-1} \circ A$ for each $n \geq 1$. Indeed, let ξ_γ be a net in M converging to $\xi \in M$. Let us denote by $\psi_\gamma = (I-\lambda_n B)^{-1} \circ A(\xi_\gamma)$. Since M is compact, up to a subnet we might assume $\psi_\gamma \to \psi$, thus, using the continuity of A and B we have

$$A(\xi_\gamma) = (I-\lambda_n B)(\psi_\gamma) \to (I-\lambda_n B)(\psi).$$

Thus $\psi = (I-\lambda_n B)^{-1} \circ A(\xi)$. We now can apply, for each $n \geq 1$, Theorem 2.2 to A and $\lambda_n B$ in order to get a fixed point u_n for $A + \lambda_n B$, i.e, there exists a u_n such that

$$u_n = \lambda_n B(u_n) + A(u_n). \tag{2.1}$$

Finally, up to a subnet we may suppose $u_n \to u$ in M. Passing the limit in (2.1) we conclude the proof of Corollary 2.3. □

Another variant of Corollary 2.3, where the compactness of M is substituted by the relative compactness of $A(M)$ is the following.

Corollary 2.4. *Let M be a convex and closed subset of a Hausdorff locally convex space X and $A, B\colon M \to X$ be continuous operators such that*

(a) $(I-B)$ *is injective and* $A(M) \subseteq (I-\lambda_n B)(M)$, $\forall n$;
(b) $A(M)$ *is relatively compact.*
(c) *If* $u = \lambda_n B(u) + A(v)$ *for some* $v \in M$ *then* $u \in M$.

Then $A+B$ has a fixed point in M.

Remark 2.5. It is worthwhile to highlight that the condition of $A(M) \subseteq (I-\lambda B)(M)$ in Theorem 2.2, Corollary 2.3 and Corollary 2.4 can be relaxed by $A(M) \subseteq (I-\lambda B)(\mathcal{D}(B))$, where $\mathcal{D}(B)$ stands for the biggest domain for which B is continuous on it.

A simple way of checking the invertibility of $(I-B)$ and the condition of $A(M) \subseteq (I-B)(M)$ is to ask that B is a contraction is the following sense.

Definition 2.6. Let $B\colon M \to X$ be an operator defined in a subset of a locally convex space X. Let $\mathcal{T}$ be a family of seminorms that define the topology in X. We say B is a $\mathcal{T}$-contraction if for each $\rho \in \mathcal{T}$ there exists a $\lambda_\rho \in (0,1)$ such that

$$\rho(B(u) - B(v)) \leq \lambda_\rho \rho(u-v).$$

Indeed when X is complete, it is a simple adaptation of the original proof of the Banach fixed point Theorem the fact that every map $B\colon X \to X$ has a unique fixed point. For more details see [7].

Let now $(X, \|\cdot\|)$ be a Banach space and let $\mathcal{T}$ be the family of seminorms $\{\rho_f(x) = |\langle f, x\rangle| \,:\, f \in X^* \text{ and } \|f\|_{X^*} \leq 1\}$. The topology generated by $\mathcal{T}$ is called the weak topology. In practical situations, often one faces the problem of solving equations of the type (1.1) in the weak topology setting. One of the advantages of this special locally convex topology is the fact that if a set M is weakly compact, then every sequentially weakly continuous map $T: M \to X$, i.e. an operator which maps weakly convergent sequences into weakly convergent sequences, is weakly continuous. This is an immediately consequence of the fact that weak sequential compactness is equivalent to weak compactness (Eberlein-Šmulian's Theorem). From this observation, the following version of Schauder fixed point Theorem holds [2].

Theorem 2.7. *Let M be a convex weakly compact subset of a Banach space X. Then every sequentially weakly continuous operator T self-mapping M has a fixed point.*

Consequently we have the following result which will be used in this form in section 5. Such a result encloses an improvement which will turn out to be crucial for establishing the existence principle for the nonlinear integral equation studied in section 5. Before stating this result we need a definition.

Definition 2.8. Let M be a closed and convex subspace of a Hausdorff locally convex space X and $A, B\colon M \to X$ continuous operators. We will denote by $\mathcal{F} = \mathcal{F}(M, A, B)$ the following set

$$\mathcal{F} := \{u \in X : u = B(u) + A(v) \text{ for some } v \in M\}.$$

Theorem 2.9. *Let M be a closed, convex subset of a Banach space X. Assume that $A, B : M \to X$ satisfies:*

(a) *A is sequentially weakly continuous;*
(b) *B is λ-contraction;*
(c) *If $u = Bu + Av$, for some $v \in M$, then $u \in M$;*
(d) *If $\{u_n\}$ is a sequence in $\mathcal{F}$ such that $u_n \rightharpoonup u$, for some $u \in M$, then $Bu_n \rightharpoonup Bu$;*
(e) *The set $\mathcal{F}$ is relatively weakly compact.*

Then $A + B$ has a fixed point in M.

Proof. Fix a point $u \in M$ and let Tu be the unique point in X such that $Tu = B \cdot Tu + Au$. By (c), we have $Tu \in M$. So that the mapping $T : M \to M$ given by $u \mapsto Tu$ is well-defined. Notice that $Tu = (I - B)^{-1}Au$, for all $u \in M$. In addition, we observe that $T(M) \subset \mathcal{F}$. We claim now that T is sequentially weakly continuous in M. Indeed, let $\{u_n\}$ be a sequence in M such that $u_n \rightharpoonup u$ in M. Since $\{Tu_n\} \subset \mathcal{F}$, the assumption (e) guarantees, up to a subsequence, that $Tu_n \rightharpoonup v$, for some $v \in M$. By (d), we have $B \cdot Tu_n \rightharpoonup Bv$. Also, from (a) it follows that $Au_n \rightharpoonup Au$ and hence the equality $Tu_n = B \cdot Tu_n + Au_n$ give us $v = Bv + Au$. By uniqueness, we conclude that $v = Tu$. This proves the claim. Take now the subset $C = \overline{\text{co}}(\mathcal{F}) \subset M$. Krein-Šmulian Theorem implies that C is a weakly compact set. Furthermore, it is easy to see that $T(C) \subset C$. Applying Theorem 2.7, we find a fixed point $u \in C$ for T. Consequently, this proves Theorem 2.9. □

Let us now state some other consequences of Theorem 2.9. The first one is the following result for reflexive Banach spaces, where closed, convex and bounded sets are weakly compacts.

Corollary 2.10. *Assume the conditions (a)-(d) of Theorem 2.9 for A and B. If M is a closed, convex and bounded subset of a reflexive Banach space, then $A + B$ has a fixed point in M.*

Next we consider the case when B is a nonexpansive mapping (or 1-Lipschitz mapping) on X, that is a mapping satisfying $\|Bu - Bv\| \leq \|u - v\|$, for all $u, v \in X$.

Corollary 2.11. *Let M be a convex and weakly compact subset of a Banach space X and let $A, B : M \to X$ be sequentially weakly continuous operators such that*

(a) *B is nonexpansive;*

(b) *If* $\lambda \in (0,1)$ *and* $u = \lambda Bu + Av$ *with* $v \in M$*, then* $u \in M$*;*

Then $A + B$ *has a fixed point in* M*.*

The above versions of Krasnoselskii fixed point Theorem generalize among others, [7], [5].

3. Local Versions of Krasnoselskii Fixed Point Theorem to One Parameter Operator Equations

Let X be a Banach space. The main goal of this section is to present some existence results for the following nonlinear equation operator on Banach spaces.

$$Au + \lambda Bu = u, \tag{3.1}$$

where $A, B : X \to X$ and $\lambda \geq 0$. In order to do so, we establish some local versions of above abstract results. We recall that a mapping $T : X \to X$ is called Lipschitzs if $\|Tu - Tv\| \leq \|T\|_{Lip}\|u - v\|$, for all $u, v \in X$, where $\|T\|_{Lip}$ denotes the Lipschitzs constant of T.

In the sequel we need of the following definition.

Definition 3.1. A mapping $T : X \to X$ is said to be expanding if $\|u\| \leq \|u - \lambda Tu\|$, holds for any $\lambda > 0$ and all $u \in X$.

Our first result concerning about equation (3.1) is as follows.

Theorem 3.2. *Let* X *be a reflexive Banach space. Assume that* $A, B : X \to X$ *are sequentially weakly continuous maps satisfying:*

(i) $A(B_R) \subset B_R$*, for some* $R > 0$*;*

(ii) B *is Lipschitzs and expanding on* X*.*

Then (3.1) is solvable for all $\lambda \geq 0$*.*

Proof. Given $\lambda \geq 0$, we define $\mathcal{A}, \mathcal{B} : X \to X$ by

$$\mathcal{A}(u) = \frac{Au + \lambda\|B\|_{Lip} \cdot u}{1 + \lambda\|B\|_{Lip}} \quad \text{and} \quad \mathcal{B}(u) = \frac{\lambda Bu}{1 + \lambda\|B\|_{Lip}}.$$

Then, $\mathcal{A}$, $\mathcal{B}$ are sequentially weakly continuous maps, $\mathcal{A}$ maps B_R into itself and $\mathcal{B}$ is a $\frac{\lambda\|B\|_{Lip}}{1+\lambda\|B\|_{Lip}}$-contraction. Now, since X is reflexive, it follows that B_R is weakly compact. Thus, by Corollary 2.3, $\mathcal{A} + \mathcal{B}$ has a fixed point $u \in B_R$. Now, one easily verifies that $Au + \lambda Bu = u$. This completes the proof.

Now we can state and prove our second result related o the solvability of equation (3.1).

Theorem 3.3. *Let* X *be a reflexive Banach space and* $B : X \to X$ *a Lipschitz mapping which is sequentially weakly continuous. Suppose that for any* $\mu > 0$, $A_\mu : X \to X$ *is a sequentially weakly continuous mapping such that*

(i) $\|A_\mu u\| \leq \mu\|u\|^p + a\|u\|^q + b$,

where $p > 1$, $0 < q < 1$ *and* $a, b > 0$*. Then there exists* $\mu^* > 0$ *such that for any* $\mu \in (0, \mu^*)$ *and any* $0 \leq \lambda \leq 1/\|B\|_{Lip}$*, the sum* $A_\mu + \lambda B$ *has a fixed point.*

Proof. We might without loss of generality suppose $B(0) = 0$. Fix $0 < \lambda < \frac{1}{\|B\|}$. Consider the ball $M = B_R(0)$ of X, where $R > 0$ is such that

$$\frac{a}{R^{1-q}} + \frac{b}{R} \leq (1 - \lambda\|B\|).$$

Now taking $\mu^* > 0$ such that

$$\mu^* R^p + aR^q + \lambda\|B\|R + b \leq R, \tag{3.2}$$

we conclude that for any $\mu \in (0, \mu^*)$, the sum $A_\mu + \lambda B$ maps M into itself. Since M is weakly compact, we can apply Theorem 2.7 to get a fixed point to $A_\mu + \lambda B$. This complete the proof. □

In our next result we consider the case where $B \equiv$ constant. The proof we shall present follows the arguments in [21].

Theorem 3.4. *Let X be a Banach space and $A : X \to X$ a compact operator such that*

(i) $\|Au\| \leq a\|u\|^p$,

where $a > 0$ and $p > 1$. Then, there exists $R > 0$ such that for any $h \in B_R$, (3.1) has a solution with $B \equiv h$.

Proof. For each $r > 0$ let δ_r be the number given by

$$\delta_r = \sup_{\|x\| \leq r} \|Ax\|.$$

From the assumption (i), we can choose $\sigma > 0$ such that

$$\inf_{0<r<\sigma} \frac{\delta_r}{r} < 1. \tag{3.3}$$

By (3.3), there exists $r > 0$ such that if $\|x\| \leq r$ we have

$$\|Ax\| \leq \delta_r < r.$$

We now define the map $T : X \to X$ by $Tx = Ax + h$. Thus, taking $0 < R < r - \delta_r$ and any $h \in B_R$, we have that

$$\|Tx\| \leq \|Ax\| + \|h\| \leq r, \tag{3.4}$$

for all $x \in B_r$. This tells us that A maps B_r into itself. By the Schauder fixed point Theorem, there exists $u \in B_r$ such that $u = Au + h$.

Remark 3.5. Other local versions of Krasnoselskii fixed point Theorem type can be found in [8].

4. An Elliptic Equation with Lack of Compactness

The main purpose of this section is to apply the above results in order to study the existence of strong solutions for a class of nonlinear elliptic problems of the form

$$-\Delta u + \lambda u = f(x, u, \mu) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with $C^{1,1}$-boundary $\partial\Omega$, λ is a real number and $f : \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$. As usual, by a strong solution of (4.1) we mean a function $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ satisfying (4.1) in the sense almost everywhere. Here, $W^{2,2}(\Omega)$ is the usual Sobolev space and $W_0^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $\|u\|_{1,2,\Omega}$. The basic idea is to reduce (4.1) to a fixed point problem in $L^2(\Omega)$ of the form

$$u = N_{f_\mu} \circ L^{-1}(u) - \lambda L^{-1}(u), \tag{4.2}$$

where N_{f_μ} is the Nemytskii operator associated to f and L^{-1} is the inverse of $L=-\Delta$. Thus, a solution of (4.2) will be a strong solution to (4.1).

For convenience of the reader, before stating our existence result to (4.1) we recall some basic facts which will used later. It is well-known that the mapping $u\mapsto Lu$ is one-to-one from $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ onto $L^2(\Omega)$. Moreover, there exists $C>0$ such that

$$\|u\|_{2,2}\le C\|Lu\|_2, \tag{4.3}$$

for all $u\in E$, see [11].

In what follows we consider the following assumption:

$$(H_p)\begin{cases} p>1/2, \text{ if } N=3,4,\\ 1<p\le N/(N-4), \text{ if } N>4.\end{cases}$$

The next basic L^p-estimate will be used in the sequel.

Proposition 4.1. *Assume (H_p). Then for any $w\in E$*

$$\|w\|_{2p}\le \gamma\|Lw\|_2, \tag{4.4}$$

where γ is a constant depending only on p, C and Ω.

Proof. This is a consequence of the Sobolev inequalities. Assume $1/2<p<\infty$. If $N=3$, then it follows from continuous inclusion $W^{2,2}(\Omega)\hookrightarrow C^0(\Omega)\hookrightarrow L^{2p}(\Omega)$, that $\|w\|_{2p}\le\gamma\|w\|_{2,2}$, for all $w\in W^{2,2}(\Omega)$. From this and by (4.3) we get (4.4). Similarly, if $N=4$, (4.4) follows from $W^{2,2}(\Omega)\hookrightarrow L^{2p}(\Omega)$ together with (4.3). Finally, in case $N>4$ and $1/2<p\le N/(N-4)$, (4.4) follows, by using the same argument as above, from $W^{2,2}(\Omega)\hookrightarrow L^{\frac{2N}{N-4}}(\Omega)\hookrightarrow L^{2p}(\Omega)$. □

The assumptions on f we shall assume is as follows.

- (f$_1$) $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ is a Caratheodory function;
- (f$_2$) there exists $\mu>0$ such that $N_{f_\mu}\circ(-\Delta)^{-1}$ maps a ball B_R of $L^2(\Omega)$ into itself.

Following the same arguments as in [6] together with Theorem 3.2, we can prove the following result.

Theorem 4.2. *Assume* (f$_1$) *and* (f$_2$). *Then problem (4.1) has a strong solution for every $\lambda\ge 0$.*

We illustrate Theorem 4.2 by means of the following simple examples.

Example 4.3. Consider the problem

$$-\Delta u+\lambda u=\mu|u|^{p-2}u+a|u|^{q-2}u+h(x) \text{ in } \Omega,\quad u=0 \text{ on } \partial\Omega, \tag{4.5}$$

where $p>2$, $a\ge 0$, $3/2\le q<2$ and $h\in L^2(\Omega)$. We now claim that if (H_{p-1}) is fulfilled, then there exists $\lambda^*>0$ such that for every $\lambda\ge-\lambda^*$, the problem (4.5) has at least one strong solution. To this end, let us suppose first that $\lambda\ge 0$. Thanks to Theorem 4.2, it is enough to show that the function $f:\Omega\times\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ defined by

$$f(x,u,\mu)=\mu|u|^{p-2}u+a|u|^{q-2}u+h(x),$$

satisfies condition (f$_2$). From the fact that (H_{p-1}) holds and by (4.4), we obtain the following estimate.

$$\|N_{f_\mu}\circ L^{-1}(u)\|_2\le \mu\gamma^{p-1}\|u\|_2^{p-1}+a\gamma^{q-1}\|u\|_2^{q-1}+\|h\|_2, \tag{4.6}$$

for all $u \in L^2(\Omega)$. Consequently, if $\|u\|_2 \le R$ then we get

$$\|N_{f_\mu} \circ L^{-1}(u)\|_2 \le \mu\gamma^{p-1}R^{p-1} + a\gamma^{q-1}R^{q-1} + \|h\|_2.$$

Now, since $0 < q-1 < 1$, we can choose $R > 0$ large enough such that

$$a\gamma^{q-1}R^{q-1} + \|h\|_2 < R.$$

Then, choosing

$$\mu^* = \frac{R - a\gamma^{q-1}R^{q-1} - \|h\|_2}{\gamma^{p-1}R^{p-1}},$$

we conclude that $\|N_{f_\mu} \circ L^{-1}(u)\|_2 \le R$, for all $\|u\|_2 \le R$ and $\mu \in (0, \mu^*)$. This implies (f_2) and proves the claim for $\lambda \ge 0$. Let now $\lambda^* = 1/\|L^{-1}\|_{\mathcal{L}(L^2(\Omega))}$ and let $\lambda \in (-\lambda^*, 0)$. Then $|\lambda| \cdot \lambda^* < 1$ and from estimate (4.6) we can apply Theorem 3.3 to get a fixed point to (4.2). Such a function will be a strong solution to (4.5). The claim is proved.

In next example we explore the case where $\mu = 1$ in (4.5).

Example 4.4. Let p and h be as above. Consider the problem

$$-\Delta u = |u|^{p-2}u + h(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{4.7}$$

In this case we can obtain a strong solution to (4.7) via fixed point Theorem since h is small enough. Indeed, define the operator A by

$$A(u) = |L^{-1}(u)|^{p-2}L^{-1}(u).$$

From inequality (4.4), it follows that A is well-defined from $L^2(\Omega)$ into itself. In addition, the same argument used in (4.6) shows that

$$\|A(u)\|_2 \le \gamma^{p-1}\|u\|_2^{p-1},$$

for all $u \in L^2(\Omega)$. In view of Theorem 3.4, problem (4.7) is solvable for h small enough.

Remark 4.5. It is worthwhile to point out that in both above examples, the power nonlinearity p might hit the critical exponent, i.e, $p = 2^* = \frac{2N}{N-2}$, for all $N \ge 3$.

5. A nonlinear integral equation: existence theory

In this section we deal with the following integral equation

$$u(t) = f(u) + \int_0^t g(s,u)ds, \quad u \in C(I,E), \tag{5.1}$$

where E is a reflexive space and $I = [0,T]$. Assume that the functions involved in equation (5.1) satisfy the following conditions

(H_1) $f : E \to E$ is sequentially weakly continuous;

(H_2) $\|f(u) - f(v)\| \le \lambda\|u - v\|$, $(\lambda < 1)$ for all $u, v \in E$;

(H_3) $\|u\| \le \|u - \big(f(u) - f(0)\big)\|$, for all $u \in E$;

(H_4) for each $t \in I$, the map $g_t = g(t,\cdot) : E \to E$ is sequentially weakly continuous;

(H_5) for each $u \in C(I,E)$, $t \mapsto g(t,u)$ is Pettis integrable on I;

(H_6) there exists $\alpha \in L^1[0,T]$ and a nondecreasing continuous function $\varphi : [0,\infty) \to (0,\infty)$ such that $\|g(s,u)\| \le \alpha(s)\varphi(\|u\|)$ for a.e $s \in [0,t]$, and all $u \in E$. Moreover, $\int_0^T \alpha(s)ds < \int_0^\infty \frac{dx}{\varphi(x)}$.

Our existence result for (5.1) is as follows.

Theorem 5.1. *Under assumptions (H_1)-(H_6), equation (5.1) has at least one solution $u \in C(I,E)$.*

Proof. Let us define the functions

$$J(z) = \int_{|f(0)|}^{z} \frac{dx}{\varphi(x)} \quad \text{and} \quad b(t) = J^{-1}\Big(\int_0^t \alpha(s)ds\Big).$$

We now define the set

$$M = \{u \in C(I,E) : \quad \|u(t)\| \le b(t) \text{ for all } t \in I\}.$$

Our strategy is to apply Theorem 2.9 in order to find a fixed point for the operator $A+B$ in M, where $A, B : M \to C(I,E)$ are defined by

$$\begin{aligned} Au(t) &= f(0) + \int_0^t g(s,u)ds, \qquad \text{and} \\ Bu(t) &= f(u(t)) - f(0). \end{aligned}$$

The proof will be given in several steps.

Step 1. M is bounded, closed and convex in $C(I,E)$.

The fact that M is bounded and closed comes directly from its definition. Let us show M is convex. Let u, v be any two points in M. Then, there holds

$$\|(1-s)u(t) + sv(t)\| \le b(t)$$

for all $t \in I$, which implies that $(1-s)u + sv \in M$, for all $s \in [0,1]$. This shows that M is convex.

Step 2. $A(M) \subseteq M$, $A(M)$ is weakly equicontinuous and $A(M)$ is relatively weakly compact.

i. Let $u \in M$ be an arbitrary point. We shall prove $Au \in M$. Fix $t \in I$ and consider $Au(t)$. Without loss of generality, we may assume that $Au(t) \neq 0$. By the Hahn-Banach Theorem there exists $\psi_t \in E^*$ with $\|\psi_t\| = 1$ such that $\langle \psi_t, Au(t)\rangle = \|Au(t)\|$. Thus,

$$\begin{aligned} \|Au(t)\| &= \langle \psi_t, Au(t)\rangle = \langle \psi_t, f(0)\rangle + \int_0^t \langle \psi_t, g(s,u)\rangle ds \\ &\le \|f(0)\| + \int_0^t \alpha(s)\varphi(\|u(s)\|)ds \\ &\le \|f(0)\| + \int_0^t \alpha(s)\varphi(b(s))ds \le b(t), \end{aligned} \tag{5.2}$$

since

$$\int_{|f(0)|}^{b(s)} \frac{dx}{\varphi(x)} = \int_0^s \alpha(x)dx.$$

Then, (5.2) implies that $A(M) \subseteq M$. Analogously one shows that,

$$\begin{aligned} \|Au(t) - Au(s)\| &\leq \int_t^s \alpha(\eta)\varphi(\|u(\eta)\|)d\eta \\ &\leq \int_t^s \alpha(\eta)\varphi(b(\eta))d\eta = \int_t^s b'(\eta)d\eta \\ &\leq |b(t) - b(s)|, \end{aligned} \tag{5.3}$$

for all $t, s \in I$. Thus it follows from (5.2) that $A(M)$ is weakly equicontinuous.

ii. Let (Au_n) be any sequence in $A(M)$. Notice that M is bounded. By reflexiveness, for each $t \in I$ the set $\{Au_n(t) : n \in \mathbb{N}\}$ is relatively weakly compact. As before, one shows that $\{Au_n : n \in \mathbb{N}\}$ is a weakly equicontinuous subset of $C(I, E)$. It follows now from the Ascoli-Arzela Theorem that (Au_n) is relatively weakly compact, which proves the third assertion of Step 2.

Step 3. A is sequentially weakly continuous.

Let (u_n) be a sequence in M such that $u_n \rightharpoonup u$ in $C(I, E)$, for some $u \in M$. Then, $u_n(s) \rightharpoonup u(s)$ in E for all $s \in I$. By assumption (H_5) one has that $g(s, u_n(s)) \rightharpoonup g(s, u(s))$ in E for all $s \in I$. The Lebesgue dominated convergence Theorem yields that $Au_n(t) \rightharpoonup Au(t)$ in E for all $t \in I$. On the other hand, it follows from (5.3) that the set $\{Au_n : n \in \mathbb{N}\}$ is a weakly equicontinuous subset of $C(I, E)$. Hence, by the Ascoli-Arzela Theorem there exists a subsequence (u_{n_j}) of (u_n) such that $Au_{n_j} \rightharpoonup v$ for some $v \in C(I, E)$. Consequently, we have that $v(t) = Au(t)$ for all $t \in I$ and hence $Au_{n_j} \rightharpoonup Au$. Now, a standard argument shows that $Au_n \rightharpoonup Au$. This proves Step 3.

Step 4. B satisfies conditions (b) and (d) of Theorem 2.9.

By (H_2) clearly we see that B is a λ-contraction in $C(I, E)$. Now, in order to verify condition (d) to B, we first remark that by combining (5.3) with (H_2), it follows that $\mathcal{F}$ is weakly equicontinuous in $C(I, E)$. So is $B(\mathcal{F})$. Let now $(u_n) \subset \mathcal{F}$ be such that $u_n \rightharpoonup u$, for some $u \in M$. Then by assumption (H_1), we obtain $Bu_n(t) \rightharpoonup Bu(t)$. Since (Bu_n) is weakly equicontinuous in $C(I, E)$ and $\|(Bu_n)(t)\| \leq \lambda\|u_n(t)\|$ holds for all $n \in \mathbb{N}$, we may apply the Ascoli-Arzela Theorem and concludes that there exists a subsequence (u_{n_j}) of (u_n) such that $Bu_{n_j} \rightharpoonup v$, for some $v \in C(I, E)$. Hence, $Bu = v$ and by standard arguments we have $Bu_n \rightharpoonup Bu$ in $C(I, E)$. This completes Step 4.

Step 5. Condition (c) of Theorem 2.9 holds.

Suppose that $u = Bu + Av$ for some $v \in M$. We will show that $u \in M$. By condition (H_3) it follows that

$$\|u(t)\| \leq \|u(t) - Bu(t)\| = \|Av(t)\|.$$

Once $v \in M$ implies $Av \in M$, we conclude $u \in M$.

Step 6. Condition (e) of Theorem 2.9 holds.

Let $(u_n) \subset \mathcal{F}$ be an arbitrary sequence. Then, (u_n) is weakly equicontinuous in $C(I, E)$. Also, one has that

$$\|u_n(t)\| \leq (1 - \lambda)^{-1} \cdot b(t),$$

for all $t \in I$, that is, for each $t \in I$ the set $\{u_n(t)\}$ is relatively weakly compact in E. Thus, invoking again the Ascoli-Arzela Theorem we obtain a subsequence

of (u_n) which converges weakly in $C(I, E)$. By the Eberlein-Šmulian Theorem, it follows that $\mathcal{F}$ is relatively weakly compact.

Theorem 2.9 now gives a fixed point for $A+B$ in M, and hence a solution to (5.1). Such a solution is, a priori, a weakly continuous curve; however since $u \in \overline{co}(\mathcal{F})$ we can conclude u is actually norm continuous. $\square$

Remark 5.2. Theorem 5.1 is a generalization of Theorem 2.4 in [19].

We complete this section by presenting a wide and illustrative class of maps f defined on the real line fulfilling condition (H_3) in Theorem 5.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$tf(t) \leq 0.$$

It is easy to check that functions satisfying the above inequality fulfill assumption (H_3) in Theorem 5.1.

6. Optimal regularity for equation (5.1) via topological methods

In this section we want to explore the optimal regularity of equation (3.1).

$$u(t) = f(u(t)) + \int_0^t g(s, u(s))ds \tag{6.1}$$

Questions related to regularity of solutions to abstract nonlinear integral equations is quite important. The importance of this question is intrinsically related to almost everywhere differentiability of curves in Banach spaces. By knowing the solution of a nonlinear integral equation is sufficiently regular we can recover the original differential equation the nonlinear integral equation models.

The conditions on equation (6.1) will be slight different. However these will not be more restrictive in practical applications. We shall assume

(C_1) $f : E \to E$ is sequentially weakly continuous;

(C_2) f is 1-Lipschitz and differentiable;

(C_3) $\|u\| \leq \|u - \big(f(u) - f(0)\big)\|$, for all $u \in E$ & $\|w\| \leq \|w - Df(u)w\|$, $\forall w, u \in E$;

(C_4) g is weak Caratheodory;

(C_5) g has polynomial growth, i.e., $\|g(s, u)\| \leq C \cdot (\|u\|^r + 1)$, for some $r \geq 0$;

(C_6) There exists $R > \|f(0)\|$ such that $C \cdot (T^r + 1) = \dfrac{R - \|f(0)\|}{R^r + 1}$.

Remark 6.1. The differentiability asked in condition (C_2) is related to one of the most important problems in nonlinear functional analysis: the "almost everywhere" differentiability of Lipschitz maps between Banach spaces. Many progress have been done in the direction of defining the right notion of "almost everywhere" for Banach spaces. (see for instance [12] and [16]). We should point out though that once the space we are working on is separable and reflexive it follows from a result due to Aronszajn, Christensen and Mankiewicz that Lipschitz functions are Gateaux differentiable outside an Aronszajn null set. See, for instance, Theorem 6.42 in [3].

Condition (C_6) is always verified by taking T small enough. Such a condition is actually a restriction on the global definition of solutions. Such a constraint appears

even in very simple versions of equation (6.1). It is possible though, to show that if $r \leq 1$ we can find a global solution to equation (6.1).

Theorem 6.2 (Optimal Regularity). *Under assumptions $(C_1)-(C_6)$ equation (6.1) has at least a Lipschitz solution. Such a solution is almost everywhere differentiable.*

Before proving Theorem 6.2 we need to develop some tools. We will start by defining a new locally convex topology to $W^{1,\infty}(I,E)$.

Definition 6.3. Let E be a Banach space and I a bounded interval in $\mathbb{R}$. Let us denote by $\mathcal{T}^n$ the family of seminoms given by $\mathcal{T}^n := \{\rho\colon W^{1,\infty}(I,E) \to \mathbb{R}_+ : \rho = |f| \text{ and } f \in [W^{1,n}(I,E)]^*\}$. We define the T-topology in $W^{1,\infty}(I,E)$ to be the locally convex topology generated by $\bigcup\limits_{n\geq 2} \mathcal{T}^n$.

Proposition 6.4. *Let E be a separable reflexive Banach space. Then*

$$\big(B_{W^{1,\infty}(I,E)}(R), T\big)$$

is metrizable.

Proof. Initially we note that for each $n \geq 2$, the space $[W^{1,n}(I,E)]^*$ is separable. The Lemma follows easily from this general metrization Theorem: *A topological space is metrizable if and only if it is regular and has a basis that is the union of at most countably many locally finite systems of open sets.* ☐

Proposition 6.5. *Let E be a separable reflexive Banach space. Then*

$$X := \big(\overline{B_{W^{1,\infty}(I,E)}(R)}, T\big)$$

is a compact space.

Proof. Let $\{u_j\} \subset X$. By the continuous embedding $W^{1,\infty}(I,E) \hookrightarrow W^{1,n}(I,E)$ we have that $\|u_j\|_{W^{1,n}} \leq |I|^{1/n}R$. Since $\big(\overline{B_{W^{1,n}(I,E)}(|I|^{1/n}R)}, \mathcal{T}^n\big)$ is a compact space, there exists a subsequence $\{u_{j_k}\}$ which converges weakly to a $u \in W^{1,n}(I,E)$. By the Cantor Diagonal Argument, we build a subsequence $\{u_{j_r}\}$ which converges weakly to u in $W^{1,n}(I,E)$ for all $n \geq 2$. This initially implies that $u_{j_r} \xrightarrow{\mathrm{T}} u$. In addition $\|u\|_{W^{1,n}} \leq |I|^{1/n}R$. Letting $n \to \infty$ leads $u \in W^{1,\infty}$ and $\|u\|_{W^{1,\infty}} \leq R$. ☐

Lemma 6.6. *Let $f : E \to E$ be a differentiable 1-Lipschitz map. Then for each $p > 1$ and $\lambda \in (0,1)$ the map $u \mapsto I - \lambda f(u)$ is a sequentially weak continuous homeomorphism between $W^{1,p}(I,E)$ onto itself.*

Proof. Let $\psi \in W^{1,p}(I,E)$ be given. By changing f by $f - f(0)$, we may assume, without loss of generality that $f(0) = 0$, Let us start by estimating $\|f(\xi)\|_{W^{1,p}(I,E)}$, for any $\xi \in W^{1,p}(I,E)$:

$$\begin{aligned} \|f(\xi)\|_{W^{1,p}} &= \|f(\xi(t))\|_{L^p} + \|\partial_t f(\xi(t))\|_{L^p} \\ &\leq \|\xi\|_{L^p} + \|(Df)(\xi(t)) \cdot \partial_t \xi(t)\|_{L^p} \\ &\leq \|\xi\|_{W^{1,p}}. \end{aligned} \tag{6.2}$$

Inequality (6.2) tells us the substitution operator $N_f : W^{1,p}(I,E) \to W^{1,p}(I,E)$ given by $N_f(\xi) := f(\xi)$ is a bounded operator. We claim N_f is a sequentially weakly continuous map. Indeed, Let $\{u_n\} \subset W^{1,p}(I,E)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(I,E)$. We know that $\|u_n\|_{W^{1,p}}$ is bounded. Hence the sequence $\{N_f(u_n)\}$ is bounded in $W^{1,p}(I,E)$. Due to the reflexibility of $W^{1,p}(I,E)$, we may assume

$N_f(u_n) \rightharpoonup v$ in $W^{1,p}(I,E)$, for some $v \in W^{1,p}(I,E)$. The idea now is to show that $v = A(u)$. The crucial information here is the continuous embedding

$$W^{1,p}(I,E) \hookrightarrow C^{\alpha}(I,E), \text{ for each } 0 \le \alpha < 1 - 1/p. \tag{6.3}$$

Thus we can conclude that for each $s \in I$, $u_n(s) \rightharpoonup u(s)$ in E. Indeed, for a fixed $\Psi \in E^*$, the map $\Psi_s \colon C(I,E) \to \mathbb{R}$ defined by $\Psi_s(u) := \Psi(u(s))$ is a continuous linear functional. Therefore we also have that

$$N_f(u_n)(s) \rightharpoonup N_f(u)(s), \forall s \in I.$$

However, the same argument as before works to show that

$$N_f(u_n)(t) \rightharpoonup v(t) \text{ in } E, \ \forall t \in I.$$

Thus, $v(t)$ has to be equal to $N_f(u)(t)$ for every $t \in I$.
Let us define $\Lambda : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ by

$$\Lambda(\xi) = \psi + \lambda f(\xi).$$

We observe that once N_f is sequentially weakly continuous, so is Λ. Moreover, the solvability of the equation

$$(I - \lambda N_f)(u) = \psi \tag{6.4}$$

is equivalent to finding a fixed point for Λ. For each $\xi \in W^{1,p}(\Omega)$, we have from the triangular inequality and inequality (6.2)

$$\begin{aligned} \|\Lambda(\xi)\|_{W^{1,p}} &\le \|\psi\|_{W^{1,p}} + \lambda \|N_f(\xi)\|_{W^{1,p}} \\ &\le \|\psi\|_{W^{1,p}} + \lambda \|\xi\|_{W^{1,p}}. \end{aligned}$$

Let us fix $M > \dfrac{\|\psi\|_{W^{1,p}}}{1-\lambda}$. For such an M we see that if $\|\xi\|_{W^{1,p}(\Omega)} \le M$, then

$$\|\Lambda(\xi)\|_{W^{1,p}} \le \|\psi\|_{W^{1,p}} + \lambda M \le M. \tag{6.5}$$

In other words, Λ maps the closed ball of radius M in $W^{1,p}(I,E)$ into itself. Let X denote $\overline{B_{W^{1,p}}(M)}$ endowed with the weak topology. So X is a compact and convex set of a locally convex space. In additional, as we pointed out before, $\Lambda : X \to X$ is a continuous map. Invoking the Leray-Schauder-Tychonoff fixed point Theorem we conclude that Λ has a fixed point which is precisely a solution to (6.4). Since $\psi \in W^{1,p}(I,E)$ was taken arbitrarily we have proven $(I - \lambda N_f)$ is onto.
We now turn our attention to uniqueness. Let us suppose that there exist $u_1,\ u_2 \in W^{1,p}(I,E)$ such that

$$(I - \lambda N_f)u_i(t) = \psi(t) \text{ for } i = 1, 2$$

Subtracting these above equations, we find

$$f(u_1(t)) - f(u_2(t)) = \lambda(u_1(t) - u_2(t)).$$

Therefore

$$\|f(u_1(t)) - f(u_2(t))\| = \lambda \|u_1(t) - u_2(t)\| \le \|u_1(t) - u_2(t)\|.$$

If $\|u_1 - u_2\| > 0$ in a set of positive measure, we would find, $\lambda \ge 1$. Hence the solution of (P) is unique.

Finally let us study the weak sequential continuity of $R_\lambda = (I - \lambda N_f)^{-1} : W^{1,p}(I,E) \to W^{1,p}(I,E)$. Suppose $R_\lambda(\psi) = u$, i.e., $(I - \lambda N_f)u = \psi$. Then

$$\begin{aligned} \|\psi\|_{W^{1,p}} &\ge \|u\|_{W^{1,p}} - \lambda \|f(u)\|_{W^{1,p}} \\ &\ge (1-\lambda)\|u\|_{W^{1,p}}. \end{aligned}$$

Writing in a better way, $\|R_\lambda(\psi)\|_{W^{1,p}} \leq \dfrac{\|\psi\|_{W^{1,p}}}{1-\lambda}$. We have just verified that R_λ is a bounded operator. Suppose $\psi_n \rightharpoonup \psi$ in $W^{1,p}(I,E)$. Let us denote by $u_n = R_\lambda(\psi_n)$. The sequence $\{u_n\}$ is bounded, therefore, up to a subsequence, we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(I,E)$. From the weak sequential continuity of $(I-\lambda N_f)$ we have

$$\psi_n = (I-\lambda N_f)(u_n) \rightharpoonup (I-\lambda N_f)(u).$$

This implies that $R_\lambda(\psi) = u$, and thus, $R_\lambda(\psi_n) \rightharpoonup R_\lambda(\psi)$ as desired. $\square$

Proof of Theorem 6.2. Define $M := \{u \in W^{1,\infty}(I,E) : \|u\|_{1,\infty} \leq R\}$, where R is given by condition (C_6). Define also,

$$A(u) := f(0) + \int_0^t g(s,u(s))ds$$

$$B(u) = f(u) - f(0)$$

The strategy is to find a fixed point in M to the operator $A+B$.

Step 1. A maps $W^{1,p}(I,E)$ into $W^{1,p/r}(I,E)$. Moreover it is sequentially weakly continuous.

Indeed, Let us estimate $\|A(u)\|_{W^{1,p/r}}$. We first deal with $\|A(u)\|_{L^{p/r}}$.

$$\|A(u)\|_{L^{p/r}} \leq \|f(0)\|_E \cdot |I|^{r/p} + \left\{\int_0^T \left\|\int_0^t g(s,u(s))ds\right\|_E^{p/r} dt\right\}^{r/p}.$$

By Jensen's inequality we obtain

$$\begin{aligned}
\left\{\int_0^T \left\|\int_0^t g(s,u(s))ds\right\|_E^{p/r} dt\right\}^{r/p} &\leq \left\{\int_0^T t^{p-1}\int_0^t \|g(s,u(s)\|_E^{p/r} ds\; dt\right\}^{r/p} \\
&\leq \left\{T^p \int_0^T C^{p/r}\big(\|u(s)\|_E^r + 1\big)^{p/r} ds\right\}^{r/p} \\
&\leq T^r C\big(\|u\|_{L^p}^r + |I|^{r/p}\big).
\end{aligned}$$

We have shown that

$$\|A(u)\|_{L^{p/r}} \leq \|f(0)\|_E \cdot T^{r/p} + T^r C\big(\|u\|_{L^p}^r + T^{r/p}\big). \tag{6.6}$$

In the same way we find

$$\|\partial_t A(u)\|_{L^{p/r}} \leq C(\|u\|_{L^p}^r + T^{r/p}). \tag{6.7}$$

Adding up the above two inequalities we end up with the following inequality

$$\|A(u)\|_{W^{1,p/r}} \leq C(T^r+1)\|u\|_{L^p}^r + T^{r/p}\left(\|f(0)\|_E + (T^r+1)C\right). \tag{6.8}$$

Let us now show he map $A\colon W^{1,p}(I,E) \to W^{1,p/r}(I,E)$ is sequentially weakly continuous. Let $\{u_n\} \subset W^{1,p}(I,E)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(I,E)$. Immediately we know that $\|u_n\|_{W^{1,p}}$ and $\sup_{n,s}\|u_n(s)\|_E$ are bounded. The last bound is due to the continuous embedding,

$$W^{1,p}(I,E) \hookrightarrow C^\alpha(I,E), \text{ for each } 0 \leq \alpha < 1 - 1/p. \tag{6.9}$$

Inequality (6.8) says in particular that A is a bounded operator. Hence the sequence $\{A(u_n)\}$ is bounded in $W^{1,p/r}(I,E)$. Due to the reflexibility of $W^{1,p/r}(I,E)$, we may assume $A(u_n) \rightharpoonup v$ in $W^{1,p/r}(I,E)$, for some $v \in W^{1,p/r}(I,E)$. The idea now is to show that $v = A(u)$. We remark, as done in the proof of Lemma 6.6, that for

each $s \in I$, $u_n(s) \rightharpoonup u(s)$ in E. Let us fix a $\Phi \in E^*$. For each $s \in I$ which the map $g(s,\cdot)\colon E \to E$ is sequentially weakly continuous, there holds

$$\Phi(g(s,u_n(s))) \longrightarrow \Phi(g(s,u(s))).$$

Moreover,

$$\begin{array}{rcl} |\Phi(g(s,u_n(s)))| & \leq & \|\Phi\|_{E^*} \cdot \|g(s,u_n(s))\|_E \\ & \leq & \|\Phi\|_{E^*} \big(C\|u_n(s)\|_E^r + A \big) \\ & \leq & \|\Phi\|_{E^*} \cdot \widetilde{C}. \end{array}$$

Follows now from Lebesgue's dominated convergence Theorem that

$$\langle\ \Phi, A(u_n)(t)\ \rangle := \Phi(f(0)) + \int_0^t \Phi(g(s,u_n(s)))ds \overset{n\to\infty}{\longrightarrow} \langle\ \Phi, A(u)(t)\ \rangle.$$

We have proven that for every $t \in I$, $A(u_n)(t) \rightharpoonup A(u)(t)$ in E. However, as we did before, the continuous embedding, $W^{1,p/r}(I,E) \hookrightarrow C^\gamma(I,E)$, for all $0 \leq \gamma < 1 - r/p$, yields

$$A(u_n)(t) \rightharpoonup v(t) \text{ in } E.$$

Thus, $v(t)$ has to be equal to $A(u)(t)$ for every $t \in I$.

Step 2. A maps M into itself.

By letting $p \to \infty$ in (6.8) we obtain that $A\colon W^{1,\infty}(I,E) \to W^{1,\infty}(I,E)$ and

$$\|A(u)\|_{W^{1,\infty}} \leq C(T^r + 1) \cdot (\|u\|_{L^\infty}^r + 1) + \|f(0)\|_E. \tag{6.10}$$

Notice that if $\|u\|_{W^{1,\infty}} \leq R$ there holds

$$\begin{array}{rcl} \|A(u)\|_{W^{1,\infty}} & \leq & C(T^r+1)\cdot(R^r+1) + \|f(0)\|_E \\ & = & R. \end{array}$$

Step 3. The map B is sequentially weakly continuous from $W^{1,p}(I,E)$ into itself. Indeed we estimate

$$\|B(u)\|_p^p = \int_0^T \|f(u(t)) - f(0)\|^p dt \leq \int_0^T \|u(t)\|^p.$$

$$\|\partial_t B(u)\|_p^p = \int_0^T \|Df(u(t))u_t(t)\|^p dt \leq \int_0^T \|u_t(t)\|^p.$$

The above inequalities show B is a bounded operator from $W^{1,p}(I,E)$ into itself. We now argue as in Step 1 to conclude the sequential weak continuity of B.

Step 4. M is T-compact and $A, B\colon M \to W^{1,\infty}(I,E)$ are T-continuous.

The fact that M is T-compact follows from proposition 6.5. Let us show A is T-continuous: Let $\{u_j\} \subset M$ be such that $u_j \overset{\mathrm{T}}{\to} u$. It means $u_j \rightharpoonup u$ in $W^{1,m}(I,E)$, for each $m \geq 2$. We have to show that $A(u_j) \rightharpoonup A(u)$ in $W^{1,n}(I,E)$ for every $n \geq 2$. To this end, let us fix $n \geq 2$ and let $m = \lceil n \cdot r \rceil$, the lowest integer bigger than $n \cdot r$. We know $u_j \rightharpoonup u$ in $W^{1,m}(I,E)$. By Step 1, $A(u_j) \rightharpoonup A(u)$ in $W^{1,m/r}(I,E)$. Once $W^{1,m/r}(I,E)$ is continuously embedded into $W^{1,n}(I,E)$, the convergence also holds in $W^{1,n}(I,E)$. We have proven A is T-continuous. A similar argument shows B is also T-continuous.

Step 5. For all $\lambda \in (0,1)$, $(I - \lambda B)$ is T-homeomorphism from $W^{1,\infty}(I,E)$ onto itself.

Let $\psi \in W^{1,\infty}(I,E)$ be given. For each $p \geq 1$, we know $\psi \in W^{1,p}(I,E)$. We then can apply Lemma 6.6 to conclude there is a unique $u \in W^{1,p}(I,E)$, which solves

$$(I - \lambda B)(u) = \psi.$$

Moreover, it follows from inequality (6.5) that

$$\|u\|_{W^{1,p}} \leq \frac{\|\psi\|_{W^{1,p}}}{1-\lambda} + \varepsilon, \ \forall \varepsilon > 0.$$

Letting $\varepsilon \to 0$ and $p \to \infty$ we conclude u in in $W^{1,\infty}(I,E)$ and moreover $\|u\|_{W^{1,\infty}} \leq \frac{\|\psi\|_{W^{1,\infty}}}{1-\lambda}$. Lemma 6.6 also provides the sequential weak continuity of $(I-\lambda B)$, $(I-\lambda B)^{-1} \colon W^{1,p}(I,E) \to W^{1,p}(I,E)$. This together with an argument like in Step 4, concludes the proof of Step 5.

Step 6. Condition (c) of main Corollary 2.3 holds.

Indeed, suppose

$$u = \lambda B(u) + A(v), \text{ for some } v \in M.$$

From condition (C_3),

$$\|u(t)\| \leq \|u(t) - [f(u(t)) - f(0)]\| = \|A(v(t)\| \leq R.$$

Analogously,

$$\|u_t(t)\| \leq \|\big(I - Df(u(t))\big)u_t(t)\| = \|g(t,v(t))\| \leq R.$$

Thus $\|u\|_{W^{1,\infty}} \leq R$.

We have verified all the hypothesis of Corollary 2.3, which assures a fixed point to $A+B$. Such a fixed point is Lipschitz since it lies in $W^{1,\infty}(I,E)$. Furthermore, since E is reflexive it has the Radon-Nikodým property. Therefore the Lipschitz solution $u \colon I \to E$ is almost everywhere differentiable. □

Remark 6.7. When we are dealing with real valued functions there is a simple way of verifying condition (C_3) in Theorem 6.2. Indeed if f is nonincreasing and satisfies $tf(t) \leq 0$, it is easy to verify that condition (C_3) is fulfilled.

7. A geometric approach to fixed point results for sum of operators on uniformly convex spaces.

In this section we are interested in a variant of equation (5.1). Indeed we shall work on the following nonlinear integral equation:

$$u(t) = f(t,u(t)) + \Phi\left(t, \int_0^t k(t,s)u(s)ds\right), \quad u \in L^p(I,E), \tag{7.1}$$

where E is a uniformly convex space, $1 < p < \infty$ and $I = [0,T]$. The strategy here is rather different from the strategy used in the previous sections. Indeed we shall explore a geometric idea to guarantee somehow the hard-to-check assumption of Theorem 2.2, i.e., condition (c). We shall develop these ideas directly to analyze the important nonlinear integral equation (7.1).

Let us point out that, a priori, equation (7.1) is more delicate than (5.1), since the nonlinearity Φ nulls the regularity property of the integral. In the study of equation (7.1), the natural assumptions on f, k and Φ are:

- (A_1) $f \colon I \times E \to E$ is a measurable family of maps satisfying $\|f(t,x) - f(t,0)\| \leq \|x\| \ \forall x \in E$.
- (A_2) $k \in L^\infty(I, L^q(0,T))$, where $\frac{1}{p} + \frac{1}{q} = 1$. We shall denote by $C := \|k\|_\infty$.

(A_3) Φ is a weak Carathéodory map satisfying
$\|\Phi(t,u)\|_E \leq G(t)\psi(\|u\|_E)$, where,

$$\begin{cases} G \in L^p(0,T) \\ \psi \in L^\infty_{\text{loc}}(0,T) \text{ and } \exists \overline{R} \text{ with } \dfrac{\|G\|_p \cdot \psi(C \cdot \overline{R})}{\overline{R} - \|f(t,0)\|_p} \leq 1. \end{cases} \tag{7.2}$$

Let us recall that a map $\Phi\colon I \times E \to E$ is said to be a weak Carathéodory map if for each $x \in E$ fixed, the map $t \mapsto \Phi(t,x)$ is measurable and for almost every $t \in I$ the map $x \mapsto \Phi(t,x)$ is sequentially weakly continuous.

Remark 7.1. As we shall see in the proof of Theorem 7.8, condition (A_1) only need to be held for $x \in B_E(\overline{R})$. In assumption (A_3) we may always assume ψ nondecreasing and everywhere defined. The existence of such a $\overline{R}$ in hypothesis (7.2) is less restrictive than natural assumptions on the asymptotic behavior of ψ.

To grapple with the difficulty of losing the regularity properties of the integral, we will need to develop a sequential weak continuity result for a class of operators acting on vector-valued Lebesgue spaces. Moreover a geometric assumption will also be needed to assure the existence of a solution to problem (7.1). Such an assumption is somehow related to monotonicity hypothesis on the operators involved in problem (7.1). Let us start by the definitions and main results involved in such a geometric condition.

Definition 7.2. Let E be a normed vector space. We define the notion of angle between two nonzero vectors $x,\ y$ as follows:

$$\alpha(x,y) := \left\| \frac{x}{\|x\|_E} - \frac{y}{\|y\|_E} \right\|_E$$

Let now E be a uniformly convex space. Its modulus of convexity, δ, is defined as

$$\sup\left\{ \left\| \frac{x+y}{2} \right\|_E : \|x\|_E = \|y\|_E = 1;\ \|x-y\|_E = \varepsilon \right\} = 1 - \delta(\varepsilon).$$

Lemma 7.3. *Let $v_1,\ v_2,\ \ldots,\ v_n$ be nonzero elements of a uniformly convex space E. Suppose $V := \sum_{i=1}^n v_i \neq 0$. Let us denote by $\alpha_i = \alpha(v_i, V)$. Then*

$$\|V\|_E \leq \sum_{i=1}^n \Big(1 - 2\delta(\alpha_i)\Big) \cdot \|v_i\|_E,$$

where δ is the modulus of convexity of E.

Proof. It follows from the definition of the modulus of convexity that for each i, running from 1 to n, we have

$$\Big\| \|V\|_E v_i + \|v_i\|_E V \Big\|_E \leq 2\big(1 - \delta(\alpha_i)\big)\|V\|_E \cdot \|v_i\|_E.$$

Summing the above inequality over i we find

$$\sum_{i=1}^n \Big\| \|V\|_E v_i + \|v_i\|_E V \Big\|_E \leq 2\sum_{i=1}^n \big(1 - \delta(\alpha_i)\big)\|V\|_E \cdot \|v_i\|_E.$$

We now apply the standard triangular inequality to the left hand side of the above inequality and end up with

$$\|V\|_E \cdot \Big(\|V\|_E + \sum_{i=1}^{n} \|v_i\|_E\Big) \leq 2\|V\|_E \sum_{i=1}^{n} \big(1 - \delta(\alpha_i)\big)\|v_i\|_E.$$

Cancelling $\|V\|_E$ out from the above and rearranging the reminder part we conclude the Lemma. $\square$

Definition 7.4. Let E be a uniformly convex space. We define $\epsilon_0 = \epsilon_0(E) > 0$ to be the smallest positive number such that whenever we write $\epsilon_0 = \epsilon_1 + \epsilon_2$ with $0 \leq \epsilon_1, \epsilon_2 \leq 2$, we have

$$\delta(\epsilon_1) + \delta(\epsilon_2) \geq 1/2. \tag{7.3}$$

Definition 7.5. Let C be a cone in a normed vector space. We define its opening as follows:

$$\theta(C) := \sup\{\alpha(x, y) : x, y \in C\}.$$

Let us relate these definitions to problem (7.1). We define $g(t, x) := f(t, x) - f(t, 0)$ and $B\colon L^p(I, E) \to L^p(I, E)$ by

$$B(u)(t) = g(t, u(t)).$$

Also we define $A\colon L^p(I, E) \to L^p(I, E)$ by

$$A(u)(t) := f(t, 0) + \Phi\left(t, \int_0^t k(t, s)u(s)ds\right).$$

The condition on B is:

(A_4) $B\colon L^p(I, E) \to L^p(I, E)$ is sequentially weakly continuous.

Remark 7.6. The sequential weak continuity of substitution operators acting on vector-valued Lebesgue spaces was studied in [17] and [18]. We should mention that in many practical applications which arise from physical models one can verify condition (A_4). It has been of large interest of the authors the study of sufficient condition to assure condition (A_4). One of the simplest technic which has contemplated many practical situations is the following: There exists a Banach space F such that E is compacted embedded into F and for some $s < 0$ the operator $B\colon L^p(I, E) \to L^p(I, E)$ extends to $B\colon W^{s,p}(I, F) \to W^{s,p}(I, F)$ in a demicontinuous fashion, where $W^{s,p}(I, F)$ is the Sobolev-Slobodeckii spaces. Indeed if the above holds, $L^p(I, E)$ is compacted embedded into $W^{s,p}(I, F)$ (see [1]). Let $u_n \rightharpoonup u$ in $L^p(I, E)$. Since B is a bounded operator, up to a subsequence we might assume that $B(u_n) \rightharpoonup v$ for some v in $L^p(I, E)$. By the compact embed $L^p(I, E) \hookrightarrow W^{s,p}(I, F)$ we know $u_n \to u$ and $B(u_n) \to v$ in $W^{s,p}(I, F)$. Finally, since $B\colon W^{s,p}(I, F) \to W^{s,p}(I, F)$ is demicontinuous, $B(u_n) \rightharpoonup B(u)$ in $W^{s,p}(I, F)$ and thus $B(u) = v$.

Our geometric condition is as follows:

(A_5) [**Monotonicity condition**] For each $u \in L^p(I, E)$,

$$\alpha(A(u), (A + B)(u)) + \alpha(B(u), (A + B)(u)) \geq \epsilon_0(L^p(I, E))$$

Remark 7.7. Condition (A_5) is easier to verify that it might seem. For instance, if $f = f(t)$ is constant, and then equation (7.1) reduces to a nonlinear generalization of the Volterra equation, condition (A_5) is immediately verified. In this case all one needs is the reflexivity of E. Another common way to verify condition (A_5) is to assure the existence of a cone C in $L^p(I, E)$ with opening $2 - \epsilon_0(L^p(I, E))$ such that $\mathcal{I}m(A) \subseteq C$ and $\mathcal{I}m(B) \subseteq -C$. Indeed, let $\zeta_A \in \mathcal{I}m(A) \setminus \{0\} \subseteq C$ and $\zeta_B \in \mathcal{I}m(B) \setminus \{0\} \subseteq (-C)$. There holds,

$$\alpha(\zeta_A, \zeta_B) := \left\| \frac{\zeta_A}{\|\zeta_A\|} - \frac{\zeta_B}{\|\zeta_B\|} \right\| = \left\| \frac{\zeta_A}{\|\zeta_A\|} + \frac{\zeta_B}{\|\zeta_B\|} - 2\frac{\zeta_B}{\|\zeta_B\|} \right\| \geq 2 - \alpha(\zeta_A, -\zeta_B) \geq \epsilon_0.$$

To conclude we recall that for any three nonzero vectors, v_1, v_2, v_3 in a normed vector space, we have, $\alpha(v_1, v_3) \leq \alpha(v_1, v_2) + \alpha(v_2, v_3)$. Therefore

$$\epsilon_0 \leq \alpha(\zeta_A, \zeta_B) \leq \alpha(\zeta_A, \zeta_A + \zeta_B) + \alpha(\zeta_B, \zeta_A + \zeta_B).$$

We now can state the main result of this section.

Theorem 7.8. *Assume (A_1)–(A_5). Then there exists a $u \in L^p(0, T, E)$ solving the nonlinear integral equation (7.1).*

Before proving Theorem 7.8 we need to develop a sequentially weak continuity result for a class of operators acting on vector-valued Lebesgue spaces. It is worthwhile to highlight that such a result has it own importance in the theoretical point of view. This is the content of what follows. The first result we shall need is the fact that weak convergence in $L^\infty(E)$ implies a.e. weak convergence in E. This result was firstly proven in [15]. His proof though makes use of lifting theory which brings an abstract flavor to it. The proof we shall present here was taken from [22], where some other consequences of this fact is explored. Such a proof seems to give more insight as to why this phenomenon should hold. Moreover, we also believe the strategy presented here might be used in more general situation to extract, almost everywhere (weak) convergence from particular weakly convergent sequences, for instance minimizing sequences in optimization problems.

Theorem 7.9. *Let E be an arbitrary Banach space and (Ω, μ) be a Radon measure space. Let u_n be a sequence in $L^\infty(\Omega, E)$. Suppose $u_n \rightharpoonup u$ in $L^\infty(\Omega, E)$. Then for μ-almost every $x \in \Omega$,*

$$u_n(x) \rightharpoonup u(x) \text{ in } E.$$

Proof. Let $\varphi \in E^*$ be fixed. For each $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, we define $\Phi_r^x \in \left[L^\infty(\Omega, E)\right]^*$ to be

$$\Phi_r^x(f) := \frac{1}{\mu(B(x, r))} \int_{B(x,r)} \varphi(f(\xi)) d\mu(\xi). \tag{7.4}$$

We verify that

$$\|\Phi_r^x\|_{[L^\infty(\Omega,E)]^*} := \sup_{f \in L^\infty(\Omega,E)\setminus\{0\}} \frac{\Phi_r^x(f)}{\|f\|_{L^\infty(E)}} \leq \|\varphi\|_{E^*}.$$

It follows therefore, from the Banach-Alaoglu Theorem, that, for a fixed $x \in \Omega$, up to a subnet, we have

$$\Phi_r^x \overset{\star}{\rightharpoonup} \Phi^x \in \left[L^\infty(\Omega, E)\right]^*.$$

Let us, hereafter, denote $v_n := u_n - u \in L^\infty(\Omega, E)$. Let Ω_n be the Lebesgue set of v_n provided by Lebesgue's differentiation Theorem . We then set $\Omega_0 = \bigcap_{n=1}^{\infty} \Omega_n$. In this way, Ω_0 has total measure and for each $x \in \Omega_0$ there holds

$$\Phi_r^x(v_n) \stackrel{r\to 0}{\longrightarrow} \varphi(v_n(x)) = \Phi^x(v_n(x)) \stackrel{n\to\infty}{\longrightarrow} 0.$$

□

In our next result we shall make use of Dunford's Theorem, which we will state for convenience.

Theorem 7.10 (Dunford)**.** *Let (Ω, Σ, μ) be a finite measure space and X be a Banach space such that both X and X^* have the Radon-Nikodým property. A subset K of $L^1(\Omega, X)$ is relatively weakly compact if and only if*

1. *K is bounded,*
2. *K is uniformly integrable, and*
3. *for each $B \in \Sigma$, the set $\{\int_B f d\mu : f \in K\}$ is relatively weakly compact.*

Lemma 7.11. *Let $p,\ q \geq 1$ and $I\colon L^p(\Omega, E) \to L^\infty(\Omega, E)$ be a continuous linear map. Let $f\colon \Omega \times E \to E$ be a weak Carathéodory map satisfying*

$$\|f(x, u)\|_E \leq A(x)\psi(\|u\|_E),$$

where $A \in L^q(\Omega)$ and $\psi \in L^\infty_{\text{loc}}(\Omega)$. Then if either $q > 1$ or $p = q = 1$, the map $\Psi := N_f \circ I\colon L^p(\Omega, E) \to L^q(\Omega, E)$ is sequentially weakly continuous.

Proof. Let us suppose $q > 1$. Let $u_n \rightharpoonup u$ in $L^p(\Omega, E)$. Since Ψ is a bounded operator and $L^q(\Omega, E)$ is reflexive, up to a subsequence, $\Psi(u_n) \rightharpoonup v \in L^q(\Omega, E)$ for some $v \in L^q(\Omega, E)$. The idea is to show that actually $v = \Psi(u)$. From Theorem 7.9, we know $I(u_n)(x) \rightharpoonup I(u)(x)$ in E for μ-a.e. $x \in \Omega$. Since f is a weak Carathéodory map, $\Psi(u_n)(x) \rightharpoonup \Psi(u)(x)$ in E for μ-a.e. $x \in \Omega$ as well. Now we shall conclude that $v = \Psi(u)$ μ-a.e. To this end we start by throwing away a set $\mathcal{A}_0$ of measure zero such that

$$F := \overline{\text{span}}\Big[v\big(\Omega \setminus \mathcal{A}_0\big) \cup \Psi(u)\big(\Omega \setminus \mathcal{A}_0\big)\Big]$$

is a separable and reflexive Banach space. The existence of such a $\mathcal{A}_0$ is due to Pettis' Theorem. Let now $\{\varphi_j\}$ be a dense sequence of continuous linear functionals in F. By Ergorov's Theorem, for each φ_j fixed, there exists a negligible set $\mathcal{A}_j$, such that $\varphi_j(v) = \varphi_j(\Psi(u))$ in $\Omega \setminus \mathcal{A}_j$. Finally we define $\mathcal{A} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. In this way $\mu(\mathcal{A}) = 0$ and by the Hahn-Banach Theorem, $v(x) = \Psi(u)(x)$ for all $x \in \Omega \setminus \mathcal{A}$.

Let us now study the case when $p = q = 1$. For simplicity, we will restrict ourselves to finite measure spaces. We shall use Dunford's Theorem. Let $u_n \rightharpoonup u$ in $L^1(\Omega, E)$. By the Eberlein-Smulian Theorem the set $K = \{u,\ u_n\}_{n=1}^\infty$ is weakly compact. Let us show $\Psi(K)$ is relatively weakly compact in $L^1(\Omega, E)$. Clearly $\Psi(K)$ is bounded, once $\|\Psi(v)\|_{L^1(\Omega,E)} \leq \|A\|_{L^1} \cdot \psi(\|I\| \cdot \|v\|_{L^1(\Omega,E)})$. The last inequality also shows $\Psi(K)$ is uniformly integrable. Since E is reflexive, we get item (3) of Durford's Theorem for free. Hence, $\Psi(K)$ is relatively weakly compact in $L^1(\Omega, E)$. Now we proceed as in the previous case. □

We now have all the ingredients to prove Theorem 7.8.

Proof of Theorem 7.8. As done before, let $g(t, x) := f(t, x) - f(t, 0)$ and $B\colon L^p(I, E) \to$

$L^p(I,E)$ be $B(u)(t) := g(t,u(t))$. We estimate

$$\begin{aligned} \|B(u)\|_p &= \left(\int_0^T \|g(t,u(t))\|_E^p dt\right)^{1/p} \\ &\leq \|u\|_p. \end{aligned} \tag{7.5}$$

Let $\mathcal{K}\colon L^p(0,T,E) \to L^\infty(0,T,E)$ be the following map

$$\mathcal{K}(u)(t) := \int_0^t k(t,\lambda)u(\lambda)d\lambda.$$

We estimate

$$\begin{aligned} \|\mathcal{K}(u)(t)\|_E &\leq \int_0^t |k(t,\lambda)| \cdot \|u(\lambda)\|_E d\lambda \\ &\leq \|k(t,\cdot)\|_q \cdot \|u\|_p \\ &\leq C\|u\|_p. \end{aligned}$$

Above inequality says that

$$\|\mathcal{K}(u)\|_\infty \leq C\|u\|_p. \tag{7.6}$$

We remind $A\colon L^p(I,E) \to L^p(I,E)$ was defined to be

$$A(u)(t) := f(t,0) + \Phi\left(t, \int_0^t k(t,s)u(s)ds\right).$$

Showing the existence of a solution to problem (7.1) is equivalent to finding a fixed point to $A+B$. Let us now estimate $\|A(u)\|_p$:

$$\begin{aligned} \|A(u)\|_p &\leq \|f(\cdot,0)\|_p + \|\Phi(t,\mathcal{K}(u)(t))\|_p \\ &\leq \|f(\cdot,0)\|_p + \|G\|_p \cdot \psi(C \cdot \|u\|_p). \end{aligned} \tag{7.7}$$

Let $M := B_{L^p(E)}(\overline{R})$, be the ball in $L^p(0,T,E)$ with radius $\overline{R}$. It follows from (7.7) that if $\|u\|_p \leq \overline{R}$ then

$$\|A(u)\|_p \leq \|f(\cdot,0)\|_p + \|G\|_p \cdot \psi(C \cdot \overline{R}) \leq \overline{R}.$$

It implies that A maps M into itself. Estimate (7.5) shows that B also maps M into itself. Moreover Lemma 7.11 says $A\colon M \to M$ is sequentially weakly continuous. We now make use of the monotonicity condition to estimate the size of $(A+B)(M)$. From Lemma 7.3,

$$\|A(u)+B(u)\| \leq 2\overline{R}\Big[1 - \big[\alpha(A(u), A(u)+B(u)) + \alpha(A(u)+B(u), B(u))\big]\Big] \leq \overline{R}.$$

Keeping in mind that M is a compact subset of a locally convex Hausdorff space, and $A+B$ is a continuous map from M into itself, we guarantee, by the Schauder-Tychonoff Theorem, the existence of a fixed point to $A+B$. $\square$

Remark 7.12. An important case of equation (7.8) is when $f \in L^\infty(I,E)$, $\|k(t,u)\| \leq C$ and $\|\Phi(t,u)\| \leq G\psi(\|u\|)$. In this case, as we have observed above, we do not need the geometric condition (A_5). By using a similar idea of section 6, we can build the locally convex topology in $L^\infty(I,E)$ given by $\bigcup_{n\geq 2} \mathcal{T}^n$, where $\mathcal{T}^n := \{\rho\colon L^\infty(I,E) \to \mathbb{R}_+ : \rho = |f| \text{ and } f \in [L^n(I,E)]^*\}$. In this topology the ball of $L^\infty(I,E)$ is compact. Furthermore we can also verify the continuity of the operators A and B with respect to this new topology. Thus, in this particular

case, we can find a bounded solution of equation (7.8). This is as much as one should expect due to the nonlinearities involved in the equation.

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